

Mapping Class Groups of Compression Bodies and 3-Manifolds

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Abstract

We analyze the mapping class group $\mathcal{H}_x(W)$ of automorphisms of the exterior boundary W of a compression body (Q, V) of dimension 3 or 4 which extend over the compression body. Here V is the interior boundary of the compression body (Q, V) . Those automorphisms which extend as automorphisms of (Q, V) rel V are called discrepant automorphisms, forming the mapping class group $\mathcal{H}_d(W)$ of discrepant automorphisms of W in Q . If $\mathcal{H}(V)$ denotes the mapping class group of V we describe a short exact sequence of mapping class groups relating $\mathcal{H}_d(W)$, $\mathcal{H}_x(W)$, and $\mathcal{H}(V)$.

For an orientable, compact, reducible 3-manifold W , there is a canonical “maximal” 4-dimensional compression body whose exterior boundary is W and whose interior boundary is the disjoint union of the irreducible summands of W . Using the canonical compression body Q , we show that the mapping class group $\mathcal{H}(W)$ of W can be identified with $\mathcal{H}_x(W)$. Thus we obtain a short exact sequence for the mapping class group of a 3-manifold, which gives the mapping class group of the disjoint union of irreducible summands as a quotient of the entire mapping class group by the group of discrepant automorphisms. The group of discrepant automorphisms is described in terms of generators, namely certain “slide” automorphisms, “spins,” and “Dehn twists” on 2-spheres.

Much of the motivation for the results in this paper comes from a research program for classifying automorphisms of compact 3-manifolds, in the spirit of the Nielsen-Thurston classification of automorphisms of surfaces.

1 Introduction

Definition 1.1. A *compression body* is a manifold triple (Q, W, V) of any dimension $n \geq 3$ constructed as $V \times I$ with 1-handles attached to $V \times 1$, and with $V = V \times 0$. Here V is a manifold, possibly with boundary, of dimension $n - 1 \geq 2$. We use the

symbol A to denote $\partial V \times I$ and W to denote $\partial Q - (V \cup \text{int}(A))$. See Figure 1. When W is closed, we can regard the compression body as a manifold pair, and even when $\partial W \neq \emptyset$, we will often denote the compression body simply as (Q, V) .

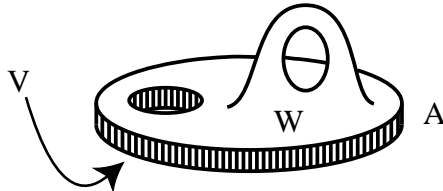


Figure 1: Compression body.

Remark: In most applications, a compression body satisfies the condition that V contains no sphere components, but we choose not to make this a part of the definition.

We note that, when $\partial V \neq \emptyset$, it is crucial in this paper that there be a product “buffer zone” A between ∂V and ∂W . Some applications of the isotopy extension theorem would fail without this buffer.

Definition 1.2. The symbol \mathcal{H} will be used to denote the mapping class group of orientation preserving automorphisms of an orientable manifold, a manifold pair, or a manifold triple. When (Q, V) is a compression body, and $\partial V = \emptyset$, $\mathcal{H}(Q, V)$ is the mapping class group of the manifold pair; when $\partial V \neq \emptyset$ by abuse of notation, we use $\mathcal{H}(Q, V)$ to denote $\mathcal{H}(Q, W, V)$.

Convention: Throughout this paper, manifolds will be compact and orientable; automorphisms will be orientation preserving.

Definition 1.3. Suppose (Q, V) is a compression body. Let $\mathcal{H}_x(W)$ denote the image of the restriction homomorphism $H(Q, V) \rightarrow \mathcal{H}(W)$, i.e. the subgroup of $\mathcal{H}(W)$ consisting of automorphisms of W which extend over Q to yield an element of $\mathcal{H}(Q, V)$. We call the elements of $\mathcal{H}_x(W)$ *extendible* automorphisms of W . Let $\mathcal{H}_d(W)$ denote the image of the restriction map $H(Q \text{ rel } V) \rightarrow \mathcal{H}(W)$, i.e. the subgroup of $\mathcal{H}_x(W)$ consisting of isotopy classes of automorphisms f of W which extend over Q such that $f|_V$ becomes the identity. We call the elements of $\mathcal{H}_d(W)$ *discrepant* automorphisms of W .

A fundamental fact well known for compression bodies of dimension 3 is that an automorphism of W which extends over a compression body Q with $\partial_e Q = W$ uniquely determines up to isotopy the restriction of the extended automorphism to $V = \partial_i Q$. In general we have a weaker statement:

Theorem 1.4. *Suppose (Q, V) is a compression body of dimension $n \geq 3$. Suppose at most one component of V is homeomorphic to a sphere S^{n-1} and suppose each component of V has universal cover either contractible or homeomorphic to S^{n-1} . If $f : (Q, V) \rightarrow (Q, V)$ is an automorphism with $f|_W$ isotopic to the identity, then:*

- (a) *$f|_V$ is homotopic to the identity. It follows that for an automorphism f of (Q, V) , $f|_W$ determines $f|_V$ up to homotopy.*
- (b) *If Q is 3-dimensional, $f|_W$ determines $f|_V$ up to isotopy, and $\mathcal{H}_x(W) \approx \mathcal{H}(Q, V)$.*
- (c) *If Q is 4-dimensional, and each component of V is irreducible, then $f|_W$ determines $f|_V$ up to isotopy.*

Suppose (Q, V) is a compression body of dimension 3 or 4 satisfying the conditions of Theorem 1.4 (b) or (c). Then there is an *eduction* homomorphism $e : \mathcal{H}_x(W) \rightarrow \mathcal{H}(V)$. For $f \in \mathcal{H}_x(W)$, we extend f to $\bar{f} : (Q, V) \rightarrow (Q, V)$, then we restrict \bar{f} to V obtaining $e(f) : V \rightarrow V$. By the theorem, e is well-defined.

Theorem 1.5. *Suppose (Q, V) is a compression body of dimension 3 or 4. Suppose at most one component of V is a sphere. In case Q has dimension 4, suppose every component of V is a 3-manifold whose universal cover is contractible or S^3 . Then the following is a short exact sequence of mapping class groups.*

$$1 \rightarrow \mathcal{H}_d(W) \rightarrow \mathcal{H}_x(W) \xrightarrow{e} \mathcal{H}(V) \rightarrow 1.$$

The first map is inclusion, the second map is eduction.

There is a *canonical 4-dimensional compression body* (Q, V) associated to a compact reducible 3-manifold W without boundary spheres, see Section 4. It has the following properties.

Proposition 1.6. *The canonical compression body (Q, V) associated to a compact reducible 3-manifold W has exterior boundary W and interior boundary the disjoint union of its irreducible summands. It is unique in the following sense: If (Q_1, W, V_1) and (Q_2, W, V_2) are two canonical compression bodies, and $v : V_1 \rightarrow V_2$ is any homeomorphism, then there is a homeomorphism $g : (Q_1, V_1) \rightarrow (Q_2, V_2)$ with $g|_{V_1} = v$.*

We will initially construct a canonical compression body using a “marking” by a certain type of system \mathcal{S} of essential spheres in W , called a *symmetric system*. The result is the *canonical compression body associated to W with the symmetric system \mathcal{S}* . The canonical compression body associated to W with \mathcal{S} is unique in a different sense: If (Q_1, W, V_1) and (Q_2, W, V_2) are canonical compression bodies associated to W with \mathcal{S} , then the identity on W extends to a homeomorphism $g : Q_1 \rightarrow Q_2$, and $g|_{V_1}$ is unique up to isotopy. Only later will we recognize the uniqueness described in Proposition 1.6 when we compare canonical compression bodies constructed from different symmetric systems of spheres.

There are well-known automorphisms of W which, as we shall see, lie in $\mathcal{H}_d(W)$ with respect to the canonical compression body Q . The most important ones are called *slides*. Briefly, a slide is an automorphism of W obtained by cutting on an essential sphere S to obtain $W|S$ (W cut on S), capping one of the duplicate boundary spheres coming from S with a ball B to obtain W' , sliding the ball along a closed curve in this capped manifold and extending the isotopy of B to the manifold, then removing B and regluing on the two boundary spheres. An *interchanging slide* is done using two separating essential spheres cutting homeomorphic manifolds from W . Cut on the two spheres S_1 and S_2 , remove the homeomorphic submanifolds, cap the two boundary spheres with two balls B_1 and B_2 , choose two paths to move B_1 to B_2 and B_2 to B_1 , extend the isotopy, remove the balls, and reglue. Interchanging slides are used to interchange prime summands. A *spin* does a “half rotation” on an $S^2 \times S^1$ summand. In addition, for each embedded essential sphere, there is an automorphism of at most order 2 called a *Dehn twist* on the sphere. If slides, Dehn twists, interchanging slides, and spins are done using spheres from a given symmetric system \mathcal{S} (or certain separating spheres associated to non-separating spheres of \mathcal{S}) then they are called *\mathcal{S} -slides*, *\mathcal{S} -Dehn twists*, *\mathcal{S} -interchanging slides*, and *\mathcal{S} -spins*. An \mathcal{S} -spin is a “half-Dehn-twist” on a separating sphere associated to a non-separating sphere of \mathcal{S} , not on a sphere of \mathcal{S} .

Proposition 1.7. *If Q is the canonical compression body associated to the compact orientable, 3-manifold W with symmetric system \mathcal{S} and $W = \partial_e Q$, $V = \partial_i Q$, then*

- (a) $\mathcal{H}_x(W) = \mathcal{H}(W)$.
- (b) *The mapping class group $\mathcal{H}_d(W)$ is generated by \mathcal{S} -slides, \mathcal{S} -Dehn twists, \mathcal{S} -slide interchanges of $S^2 \times S^1$ summands, and \mathcal{S} -spins.*

Proposition 1.7 (b) is a version of a standard result due to E. César de Sá [6], see also M. Scharlemann in Appendix A of [3], and D. McCullough in [12] (p. 69).

From Proposition 1.7 and Theorem 1.5 we obtain:

Theorem 1.8. *If W is a compact, orientable, reducible 3-manifold, and V is the disjoint union of its irreducible summands, then the following sequence is exact:*

$$1 \rightarrow \mathcal{H}_d(W) \rightarrow \mathcal{H}(W) \xrightarrow{e} \mathcal{H}(V) \rightarrow 1.$$

Here $\mathcal{H}_d(W)$ is the group of discrepant automorphisms for a canonical compression body, the first map is inclusion, and the second map is eduction with respect to a canonical compression body.

Using automorphisms of compression bodies, one can reinterpret some other useful results. Suppose W is an m -manifold ($m = 2$ or $m = 3$) with some sphere boundary components. Let V_0 denote the manifold obtained by capping the sphere boundary components with balls and let P denote the union of capping balls in V_0 . There is

another appropriately chosen canonical compression body (Q, V) for W , such that V is the disjoint union $V_0 \sqcup P$. With respect to this compression body we have a group of discrepant automorphisms, $\mathcal{H}_d(W)$. We say the pair (V_0, P) is a *spotted manifold*. The mapping class group of this spotted manifold, as a pair is the same as $\mathcal{H}(W)$. The following result probably exists in the literature in some form, and can be proven more directly, and with other terminology. For example, a special case is mentioned in [1]. The reason for including it here is to make an analogy with Theorem 1.5. Let $f : W \rightarrow W$ be an automorphism. Then f clearly induces a *capped automorphism* $f_c : V_0 \rightarrow V_0$. Simply extend f to the balls of P to obtain f_c . Then f_c is determined up to isotopy by f .

Theorem 1.9. *With W and (Q, V) as above, and V_0 a connected surface or a connected irreducible 3-manifold, there is an exact sequence*

$$1 \rightarrow \mathcal{H}_d(W) \rightarrow \mathcal{H}(W) \xrightarrow{e} \mathcal{H}(V_0) \times \mathcal{H}(P) \rightarrow 1.$$

$\mathcal{H}(P)$ is the group of permutations of the balls of P ; $\mathcal{H}(V_0)$ is the mapping class group of the capped manifold. The group $\mathcal{H}_d(W)$ of discrepant automorphisms is the subgroup of automorphisms f of W which map each boundary sphere to itself such that the capped automorphism f_c is isotopic to the identity. The map $\mathcal{H}_d(W) \rightarrow \mathcal{H}(W)$ is inclusion. The eduction map $e : \mathcal{H}(W) \rightarrow \mathcal{H}(V_0) \times \mathcal{H}(P)$ takes $f \in \mathcal{H}(W)$ to (f_c, ρ) where ρ is the permutation of P induced by f .

In Section 3 we prove a lemma about mapping class groups of triples. This is the basis for the results about mapping class groups of compression bodies. Let M be an n -manifold, and let W and V be disjoint compact submanifolds of any (possibly different) dimensions $\leq n$. The submanifolds W and V need not be properly embedded (in the sense that $\partial W \hookrightarrow \partial M$) but they must be submanifolds for which the isotopy extension theorem applies, for example smooth submanifolds with respect to some smooth structure. In this more general setting, we make some of the same definitions as in Definition 1.3, as well as some new ones.

Definitions 1.10. Let (M, W, V) be a triple as above. We define $\mathcal{H}_x(V)$, the group of *extendible automorphisms* of V , as the image of the restriction map $\mathcal{H}(M, W, V) \rightarrow \mathcal{H}(V)$. Similarly, we define $\mathcal{H}_x(W, V)$, the group of *extendible automorphisms* of the pair (W, V) , as the image of the restriction map $\mathcal{H}(M, W, V) \rightarrow \mathcal{H}(W, V)$, and the group $\mathcal{H}_x(W)$ of *extendible automorphisms* of W as the image of the restriction homomorphism $\mathcal{H}(M, W, V) \rightarrow \mathcal{H}(W)$. The group $\mathcal{H}_d(W)$ of *discrepant* automorphisms of W is the image of the restriction map $\mathcal{H}((M, W) \text{ rel } V) \rightarrow \mathcal{H}(W)$.

Lemma 1.11. *Suppose (M, W, V) is a triple as above. The following is a short exact sequence of mapping class groups.*

$$1 \rightarrow \mathcal{H}_d(W) \rightarrow \mathcal{H}_x(W, V) \rightarrow \mathcal{H}_x(V) \rightarrow 1.$$

The first map is inclusion of an element of $\mathcal{H}_d(W)$ extended to (W, V) by the identity on V , the second map is restriction.

Let M be an irreducible, ∂ -reducible 3-manifold. There is a 3-dimensional characteristic compression body embedded in M and containing ∂M as its exterior boundary, first described by Francis Bonahon, which is unique up to isotopy, see [3], or see [13], [5]. Removing this from M yields an irreducible ∂ -irreducible 3-manifold. The following is an application of Lemma 1.11 which describes how mapping class groups interact with the characteristic compression body.

Theorem 1.12. *Let M be an irreducible, ∂ -reducible 3-manifold with characteristic compression body Q having exterior boundary $W = \partial M$ and interior boundary V . Cutting M on V yields Q and an irreducible, ∂ -irreducible manifold \hat{M} . There is an exact sequence*

$$1 \rightarrow \mathcal{H}_d(W) \rightarrow \mathcal{H}_x(W) \rightarrow \mathcal{H}_x(V) \rightarrow 1,$$

where $\mathcal{H}_d(W)$ denotes the usual group of discrepant automorphisms for Q , $\mathcal{H}_x(W)$ denotes the group of automorphisms which extend to M and $\mathcal{H}_x(V)$ denotes the automorphisms of V which extend to \hat{M} .

To a large degree, this paper was motivated by a research program of the author and Leonardo N. Carvalho, to classify automorphisms of compact 3-manifolds in the spirit of the Nielsen-Thurston classification. This began with a paper of the author classifying automorphisms of 3-dimensional handlebodies and compression bodies, see [13], and continued with a paper of Carvalho, [4]. Recently, Carvalho and the author completed a paper giving a classification of automorphisms for compact 3-manifolds, see [5]. At this writing, the paper [5] needs to be revised to make it consistent with this paper. In the case of a reducible manifold W , the classification applies to elements of $\mathcal{H}_d(W)$ and $\mathcal{H}(V)$, where V is the disjoint union of the irreducible summands of W , as above. The elements of $\mathcal{H}_d(W)$ are represented by automorphisms of W each having support in a connected sum embedded in W of $S^2 \times S^1$'s and handlebodies. It is these automorphisms, above all, which are remarkably difficult to understand from the point of view of finding dynamically nice representatives. The theory developed so far describes these in terms of automorphisms of compression bodies and handlebodies. Throughout the theory, both in [13] and in [5], the notion of a “spotted manifold” arises as an irritating technical detail, which we point out, but do not fully explain. Theorem 1.9 gives some explanation of the phenomenon.

I thank Leonardo N. Carvalho and Allen Hatcher for helpful discussions related to this paper.

2 Proof of Theorem 1.4.

Proof of Theorem 1.4. Suppose (Q, V) is a compression body of dimension $n \geq 3$. We suppose that every component V_i of V has universal cover either contractible or homeomorphic to S^{n-1} . We suppose also that at most one component of V is homeomorphic to a sphere. A compression body structure for Q is defined by the system \mathcal{E} of $(n-1)$ -balls together with a product structure on some components of $Q|\mathcal{E}$, and an identification of each of the remaining components of $Q|\mathcal{E}$ with the n -ball. As elsewhere in this paper, we choose a particular type of compression body structure corresponding to a system of balls \mathcal{E} such that exactly one duplicate E'_i of a ball E_i of \mathcal{E} appears on $V_i \times 1$ for each product component of $V_i \times I$ of $Q|\mathcal{E}$. In terms of handles, this means we are regarding Q as being obtained from n -balls and products by attaching just one 1-handle to each $V_i \times I$. For each component V_i of V we have an embedding $V_i \times I \hookrightarrow Q$.

Suppose $f : (Q, V) \rightarrow (Q, V)$ is the identity on W . We will eventually show that for every component V_0 of V , $f|_{V_0}$ is homotopic to the identity on V_0 (via a homotopy of pairs preserving the boundary). If V contains a single sphere component and V_0 is this sphere component, then clearly $f(V_0) = V_0$ and $f|_{V_0}$ is homotopic to the identity, since every degree 1 map of S^{n-1} is homotopic to the identity.

If V_0 is not a sphere, we will begin by showing that $f(V_0) = V_0$. Consider the case where $\pi_1(V_0)$ is non-trivial. Clearly $\pi_1(Q)$ can be expressed as a free product of the $\pi_1(V_i)$'s and some \mathbb{Z} 's. If f non-trivially permuted V_i 's homeomorphic to V_0 , the induced map $f_* : \pi_1(Q) \rightarrow \pi_1(Q)$ would not be the identity, since it would conjugate factors of the free product. On the other hand f_* must be the identity, since the inclusion $j : W \rightarrow Q$ induces a surjection on π_1 and $f : W \rightarrow W$ induces the identity on π_1 . Next we consider the case that V_0 is contractible. Then $\partial V_0 \neq \emptyset$. In this case also, we see that $f(V_0) = V_0$, since f is the identity on $(\partial V_0 \times 1) \subset \partial W$, and so $f(\partial V_0) = \partial V_0$. From our assumptions, it is always the case that V_0 is contractible, is a sphere, or has non-trivial π_1 . We conclude that $f(V_0) = V_0$ in all cases.

It remains to show that if V_0 is not a sphere, then $f|_{V_0}$ is homotopic to the identity. When Q has dimension 3, this is a standard result, see for example [13], [10]. The method of proof, using “vertical essential surfaces” in $V_0 \times I$, depends on the fact that V_0 is not a sphere. In fact, the method shows that $f|_{V_0}$ is isotopic to the identity, which proves the theorem in this dimension.

Finally now it remains to show that $f|_{V_0}$ is homotopic to the identity when $n \geq 4$, and V_0 is not a sphere. To do this, we will work in a cover of Q . We may assume V_0 is either finitely covered by the sphere, but not the sphere, or it has a contractible universal cover. We will denote by \tilde{Q}_0 the cover of Q corresponding to the subgroup $\pi_1(V_0)$ in $\pi_1(Q)$. The compression body Q can be constructed from a disjoint union of connected product components and balls, joined by 1-handles. Covering space theory then shows that the cover \tilde{Q}_0 can also be expressed as a disjoint union of connected products and balls, joined by 1-handles. The balls and 1-handles are lifts of the balls

and 1-handles used to construct Q . The products involved are: one product of the form $V_0 \times I$, products of the form $S^{n-1} \times I$, and products of the form $U \times I$, where U is contractible, these being connected components of the preimages of the connected products used to construct Q . There is a distinguished component of the preimage of V_0 in \tilde{Q}_0 which is homeomorphic to V_0 ; we denote it V_0 rather than \tilde{V}_0 , since the latter would probably be interpreted as the entire preimage of V_0 in \tilde{Q}_0 . If V_0 has non-trivial fundamental group, then this is the only component of the preimage of V_0 homeomorphic to V_0 , otherwise it is distinguished simply by the construction of the cover corresponding to $\pi_1(V_0)$, via choices of base points. Now let \check{Q}_0 denote the manifold obtained from \tilde{Q}_0 by capping all S^{n-1} boundary spheres by n -balls. Evidently, \check{Q}_0 has the homotopy type of V_0 . This can be seen, for example, by noting that \check{Q}_0 is obtained from $V_0 \times I$ and a collection of n -balls by using 1-handles to attach universal covers of $V_i \times I$, $i \neq 0$. The universal covers of $V_i \times I$ have the form $S^{n-1} \times I$ or $U \times I$ where U is contractible. When we cap the boundary spheres of \check{Q}_0 to obtain \check{Q}_0 , the $(S^{n-1} \times I)$'s are capped and become balls.

Having described the cover \tilde{Q}_0 , and the capped cover \check{Q}_0 , we begin the argument for showing that if $f : (Q, V) \rightarrow (Q, V)$ is an automorphism which extends the identity on W , and V_0 is a non-sphere component of V , then $f|_{V_0}$ is homotopic to the identity. We have chosen a compression body structure for Q in terms of \mathcal{E} , the product $V \times I$, and the collection of balls of $Q|\mathcal{E}$. Using the automorphism f we obtain another compression body structure: $Q|f(\mathcal{E})$ contains the product $f(V \times I)$, which includes $f(V_0 \times I)$ as a component. Recall that we chose \mathcal{E} such that exactly one duplicate E'_0 of exactly one ball E_0 of \mathcal{E} appears on $V_0 \times I$. We can lift the inclusion map $i : V_0 \times I \rightarrow Q$ to an embedding $\tilde{i} : V_0 \times I \hookrightarrow \tilde{Q}_0 \subset \check{Q}_0$. Again we abuse notation by denoting the image product as $V_0 \times I$, and we use E_0 to denote the lift of E_0 as well. Similarly we lift $f \circ i : V_0 \times I \rightarrow Q$ to an embedding $\widetilde{f \circ i} : V_0 \times I \rightarrow \tilde{Q}_0 \subset \check{Q}_0$.

Claim 2.1. *The lifts \tilde{i} and $\widetilde{f \circ i}$ coincide on $(V_0 \times 1) - \text{int}(E_0)$. In other words, \tilde{f} is the identity on $(V_0 \times 1) - \text{int}(E_0)$.*

Proof of Claim. We consider two cases:

Case 1: $\partial V_0 \neq \emptyset$. The lift \tilde{f} of f must take $\partial(V_0 \times I) - \text{int}(E_0)$ in \tilde{Q}_0 to itself by uniqueness of lifting, since in this case $\partial(V_0 \times I) - \text{int}(E_0)$ is connected.

Case 2: V_0 has nontrivial fundamental group. In this case, lifting $f \circ i$ and i , we see that $\widetilde{f \circ i}(V_0 \times 1 - E_0)$ and $\tilde{i}(V_0 \times 1 - E_0)$ are components in the cover \tilde{Q}_0 of the preimage of $V_0 \times 1 - E_0 \subset W \subset Q$. From the description of the cover \tilde{Q}_0 , there is only one such component homeomorphic to $V_0 - E_0$, hence both lifts must map $(V_0 \times 1) - E_0$ to the same component, and the lifts $\widetilde{f \circ i}$ and \tilde{i} must coincide on $(V_0 \times 1) - \text{int}(E_0)$ by uniqueness of lifting.

These two cases cover all the possibilities for V_0 , for if V_0 has trivial fundamental group, then V_0 has a contractible cover by assumption, since we are ruling out the case V_0 a sphere. But this implies the boundary is non-empty, and we apply Case 1. Otherwise, we apply Case 2.

This completes the proof of the claim. \square

Following our convention of writing $V_0 \times I$ for $\tilde{i}(V_0 \times I)$, we now conclude that though in general $\tilde{i}|_{E_0} \neq \widetilde{f \circ i}|_{E_0}$, these lifts are equal on ∂E_0 .

Claim 2.2. *The map $\widetilde{f \circ i}|_{E_0}$ is homotopic rel boundary to $\tilde{i}|_{E_0}$ in \check{Q}_0 .*

Proof of Claim. We consider two cases; in Case 1 the universal cover of V_0 is contractible and in Case 2 the universal cover of V_0 is S^{n-1} . Let $C \hookrightarrow \tilde{W}$ denote ∂E_0 , an $(n-2)$ -sphere, and \tilde{W}_0 denote the preimage of W in $\check{Q}_0 \subset \check{Q}_0$, and choose a base point w_0 in C .

Case 1: The universal cover of V_0 is contractible. In this case \check{Q}_0 deformation retracts to $V_0 \subset V_0 \times I$ and the universal cover $\tilde{\check{Q}}_0$ of \check{Q}_0 is contractible, by the discussion above. Hence all higher homotopy groups of \check{Q}_0 are trivial. Using a trivial application of the long exact sequence for relative homotopy groups we obtain:

$$\cdots \pi_{n-1}(\check{Q}_0, w_0) \rightarrow \pi_{n-1}(\check{Q}_0, C, w_0) \rightarrow \pi_{n-2}(C, w_0) \rightarrow \pi_{n-2}(\check{Q}_0, w_0) \cdots$$

or

$$\cdots 0 \rightarrow \pi_{n-1}(\check{Q}_0, C, w_0) \rightarrow \mathbb{Z} \rightarrow 0 \cdots,$$

which implies that all maps of $(n-1)$ -balls to (\check{Q}_0, C) which are degree one on the boundary are homotopic.

Case 2: The universal cover of V_0 is S^{n-1} (but V_0 is not homeomorphic to S^{n-1}). In this case again \check{Q}_0 deformation retracts to $V_0 \subset V_0 \times I \subset \check{Q}_0$ by the discussion above. The universal cover $\tilde{\check{Q}}_0$ of \check{Q}_0 is homotopy equivalent to S^{n-1} , and it has the structure of $S^{n-1} \times I$ attached by 1-handles to products of the form $U \times I$ with U contractible, and n -balls, some of the n -balls coming from capped preimages of $V_i \times I$ where V_i is covered by S^{n-1} . Thus the higher homotopy groups of \check{Q}_0 are the same as those of $\tilde{\check{Q}}_0$ or S^{n-1} . Again using an application of the long exact sequence for relative homotopy groups we obtain:

$$\cdots \pi_{n-1}(C, w_0) \rightarrow \pi_{n-1}(\check{Q}_0, w_0) \rightarrow \pi_{n-1}(\check{Q}_0, C, w_0) \rightarrow \pi_{n-2}(C, w_0) \rightarrow \pi_{n-2}(\check{Q}_0, w_0) \cdots$$

Here, $\pi_{n-1}(C, w_0)$ need not be trivial, but C bounds a ball in \check{Q}_0 , hence the image of $\pi_{n-1}(C, w_0)$ in $\pi_{n-1}(\check{Q}_0, w_0)$ is trivial. On the other hand, $\pi_{n-2}(\check{Q}_0, w_0) = 0$ since \check{Q}_0 has the homotopy type of S^{n-1} . Thus we obtain the sequence:

$$\cdots 0 \rightarrow \mathbb{Z} \rightarrow \pi_{n-1}(\check{Q}_0, C, w_0) \rightarrow \mathbb{Z} \rightarrow 0 \cdots$$

This sequence splits, since we can define a homomorphism

$\pi_{n-2}(C, w_0) \rightarrow \pi_{n-1}(\check{Q}_0, C, w_0)$ which takes the generator $[C]$ to the class $[E_0]$, and the composition with the boundary operator is the identity. Thus $\pi_{n-1}(\check{Q}_0, C, w_0) \approx \mathbb{Z} \oplus \mathbb{Z}$. At first glance it appears that the lift \tilde{f} of f could take the class $\beta = [\tilde{i}|_{E_0}]$ to a different class obtained by adding a multiple of the image of the generator of $\pi_{n-1}(\check{Q}_0, w_0)$. We will show this is impossible.

If the universal cover S^{n-1} of V_0 is a degree s finite cover, then we observe that the generator α of $\pi_{n-1}(\check{Q}_0, w_0)$ is represented by a degree s covering map $\varphi_1 : S^{n-1} \rightarrow \tilde{i}(V_0 \times 1)$, whose image includes s copies of E_0 . Specifically, φ_1 is the covering map $\tilde{Q}_0 \rightarrow \check{Q}_0$ restricted to the lift to \tilde{Q}_0 of $V_0 \times 1 = \tilde{i}(V_0 \times 1)$. The map φ_1 is homotopic to a similar map $\varphi_0 : S^{n-1} \rightarrow \tilde{i}(V_0 \times 0)$. Since \tilde{f} preserves orientation on $V_0 \times 0$, it takes the homotopy class of φ_0 to itself, so $\tilde{f}_*(\alpha) = \alpha$ in $\pi_{n-1}(\check{Q}_0)$. It follows that \tilde{f} also preserves the homotopy class of φ_1 viewed as an element of $\pi_{n-1}(\check{Q}_0, C, w_0)$, since $\tilde{f}(C) = C$. We denote the image of α in the relative homotopy group by the same symbol α . The preimage $\varphi_1^{-1}(E_0)$ consists of s discs. Corresponding to the decomposition of S^{n-1} into these s discs and the complement X of their union, there is an equation in $\pi_{n-1}(\check{Q}_0, C, w_0)$: $\alpha = x + s\beta$, where x is represented by $\varphi_1|_X$ and $\beta = [\tilde{i}|_{E_0}]$. To interpret $\varphi_1|_X$ as an element of relative π_{n-1} , we can use the relative Hurewicz theorem, which says that $\pi_{n-1}(\check{Q}_0, C, w_0) \approx H_{n-1}(\check{Q}_0, C)$.

Applying \tilde{f} to the equation $\alpha = x + s\beta$, we obtain the equation $\tilde{f}_*(\alpha) = \tilde{f}_*(x) + s\gamma$, where $\gamma = \tilde{f}_*(\beta) = [f \circ i|_{E_0}]$. Since $\tilde{f}_*(\alpha) = \alpha$, and also $\tilde{f}_*(x) = x$ since \tilde{f} is the identity on $(V_0 \times 1) - E_0$, we obtain the equation $\alpha = x + s\gamma$. Combining with the earlier equation $\alpha = x + s\beta$, we obtain $s\beta = s\gamma$. Since this holds in the torsion free group $\mathbb{Z} \oplus \mathbb{Z}$, it follows that $\beta = \gamma$.

This completes the proof of the claim. \square

From the claim, we now know that $\widetilde{f \circ i|_{E_0}}$ is homotopic to $\tilde{i}|_{E_0}$. We perform the homotopy rel boundary on $\widetilde{f \circ i|_E}$ and extend to $V_0 \times I$ rel $[\partial(V_0 \times I) - E_0]$. We obtain a map $\tilde{i}' : V_0 \times I \rightarrow \check{Q}$ homotopic to $\widetilde{f \circ i}$ rel $(\partial(V_0 \times I) - E_0)$ such that $\tilde{i} = \tilde{i}'$ on $V_0 \times 1$.

Pasting the two maps \tilde{i} and \tilde{i}' so that the domain is two copies of $V_0 \times I$ doubled on $V_0 \times 1$, we obtain a new map $H : V_0 \times I \rightarrow \check{Q}_0$ with $H|_{V_0 \times 0} = \text{id}$ and $H|_{V_0 \times 1} = \tilde{f}|_{V_0 \times 0}$.

Finally H can be homotoped rel $(V_0 \times \partial I)$ to $V_0 \subset \check{Q}_0$ by applying a deformation retraction from \check{Q}_0 to $V_0 = V_0 \times 0$. After this homotopy of H , we have $H : V_0 \times I \rightarrow V_0$ a homotopy in V_0 from the identity to f . In case $\partial V_0 \neq \emptyset$ in the above argument (for homotoping H to V_0), one must perform the homotopy as a homotopy of pairs, or first homotope $H|_{\partial V_0 \times I}$ to ∂V_0 . Whenever $\partial V_0 \neq \emptyset$, our construction gives a homotopy of pairs $(f, \partial f)$ on $(V_0, \partial V_0)$.

For Q of dimension 4, with components of V irreducible, Grigori Perelman's proof of Thurston's Geometrization Conjecture shows that the universal cover of each closed component of V is homeomorphic to \mathbb{R}^3 or S^3 . Components of V which have non-empty boundary are Haken, possibly ∂ -reducible, and also have contractible universal

covers. By the first part of the lemma, it follows that $f|_V$ is uniquely determined up to homotopy. But it is known that automorphisms of such 3-manifolds which are homotopic (homotopic as pairs) are isotopic, see [16], [9], [2], [15], [7], [8]. It follows that $f|_V$ is determined by $f|_W$ up to isotopy. \square

3 Proof of Theorem 1.5.

We begin with a basic lemma which is a consequence of the isotopy extension theorem.

Let M be an n -manifold, and let W and V be disjoint compact submanifolds of any (possibly different) dimensions $\leq n$. The submanifolds W and V need not be properly embedded but they must be submanifolds for which isotopy extension holds, e.g. smooth submanifolds. Various mapping class groups of the triple (M, W, V) were defined in the introduction, before the statement of Lemma 1.11, which we now prove.

Proof of Lemma 1.11. We need to prove the exactness of the sequence

$$1 \rightarrow \mathcal{H}_d(W) \rightarrow \mathcal{H}_x(W, V) \rightarrow \mathcal{H}_x(V) \rightarrow 1,$$

where the first map is inclusion of an element of $\mathcal{H}_d(W)$ extended to (W, V) by the identity on V , the second map is restriction. The restriction map takes the group $\mathcal{H}_x(W, V)$ to $\mathcal{H}_x(V)$ and is clearly surjective. It remains to show that the kernel of this restriction map is $\mathcal{H}_d(W)$. If $(f, g) \in \mathcal{H}_x(W, V)$ restricts to the identity on $\mathcal{H}_x(V)$, then g is isotopic to the identity. By isotopy extension, we can assume it actually is the identity. Thus the kernel consists of elements $(f, \text{id}) \in \mathcal{H}_x(W, V)$ such that (f, id) extends over M , which is the definition of $\mathcal{H}_d(W)$. \square

We note that $\mathcal{H}(W, V) \approx \mathcal{H}(W) \times \mathcal{H}(V)$ is a product, but this does not show that the short exact sequence of Lemma 1.11 splits, and that the group $\mathcal{H}_x(W, V)$ is a product.

Proof of Theorem 1.5. We will use Lemma 1.11. We apply the theorem to the triple (Q, V, W) , obtaining the exact sequence

$$1 \rightarrow \mathcal{H}_d(W) \rightarrow \mathcal{H}_x(W, V) \rightarrow \mathcal{H}_x(V) \rightarrow 1.$$

Given an element $f \in \mathcal{H}_x(W, V)$, this extends to an automorphism of (Q, V) , and by Theorem 1.4, $f|_V$ is determined up to isotopy, so $H_x(W, V)$ can be replaced by $H_x(W)$.

To see that $H_x(V)$ in the sequence can be replaced by $\mathcal{H}(V)$, we must show that every element of $\mathcal{H}(V)$ is realized as the restriction of an automorphism of (Q, V) . We begin by supposing f is an automorphism of V . Observe that (Q, V) can be represented as a (connected) handlebody H attached to $V \times 1 \subset V \times I$ by 1-handles, with one 1-handle connecting H to each component of $V \times 1$. In other words, if V has c components, we obtain Q by identifying c disjointly embedded balls in ∂H with

c balls in $V \times 1$, one in each component of $V \times 1$. Denote the set, or union, of balls in ∂H by \mathcal{B}_0 and the set, or union, of balls in $V \times 1$ by \mathcal{B}_1 . There is a homeomorphism $\phi : \mathcal{B}_0 \rightarrow \mathcal{B}_1$ which we use as a glueing map. Now $(f \times \text{id})(\mathcal{B}_1)$ is another collection of balls in $V \times 1$, again with the property that there is exactly one ball in each component of $V \times 1$. We can then isotope $f \times \text{id}$ to obtain $F : V \times I \rightarrow V \times I$ such that $F(\mathcal{B}_1) = \mathcal{B}_1$. Then F permutes the balls of \mathcal{B}_1 , and $\phi^{-1} \circ F \circ \phi$ permutes the balls of \mathcal{B}_0 . We use isotopy extension to obtain $G : H \rightarrow H$ with $G|_{\mathcal{B}_0} = \phi^{-1} \circ F \circ \phi$. Finally, we can combine F and G by identifying $F(\mathcal{B}_1)$ with $G(\mathcal{B}_0)$ to obtain an automorphism $g : (Q, V) \rightarrow (Q, V)$ with $g|_V = f$. \square

Using ideas in the above proof, we can also prove Theorem 1.12.

Proof of Theorem 1.12. We apply Lemma 1.11. Letting $W = \partial M = \partial_e Q$ and $V = \partial_i Q$, where Q is the characteristic compression body for M . Thus we know that the sequence

$$1 \rightarrow \mathcal{H}_d(W) \rightarrow \mathcal{H}_x(W, V) \rightarrow \mathcal{H}_x(V) \rightarrow 1$$

is exact.

In the statement of the theorem, $\mathcal{H}_x(V)$ is defined to be the group of automorphisms of V which extend to \hat{M} , whereas in Lemma 1.11, it is defined to be the group of automorphisms of V which extend to maps of the triple (M, W, V) . In fact, these are the same, since any automorphism of V also extends to Q , as we showed in the above proof of Theorem 1.5.

To verify the exactness of the sequence in the statement, it remains to show that $\mathcal{H}_x(W, V)$ is the same as $\mathcal{H}_x(W)$. To obtain an isomorphism $\phi : \mathcal{H}_x(W, V) \rightarrow \mathcal{H}_x(W)$, we simply restrict to W , $\phi(h) = h|_W$. Given an automorphism f of $\mathcal{H}_x(W)$ we know there is an automorphism g of M which extends f . Since the characteristic compression body is unique up to isotopy, we can, using isotopy extension, replace g by an isotopic automorphism preserving the pair (W, V) , and thus preserving Q , then restrict g to (W, V) to obtain an automorphism $\psi(f)$ of the pair. This is well-defined by the well-known case of Theorem 1.4; namely, for an automorphism of a 3-dimensional compression body, the restriction to the exterior boundary determines the restriction to the interior boundary up to isotopy. Clearly $\phi \circ \psi$ is the identity. Now using Theorem 1.4 again, we can check that $\psi \circ \phi$ is the identity. \square

4 The Canonical Compression Body for a 3-Manifold.

Definitions 4.1. Let W be a compact 3-manifold with no sphere boundary components. A *symmetric system* \mathcal{S} of disjointly embedded essential spheres is a system with the property that $W|\mathcal{S}$ (W cut on \mathcal{S}) consists of one ball-with-holes, \hat{V}_* , and other components \hat{V}_i , $i = 1, \dots, k$, bounded by separating spheres S_i of \mathcal{S} , each of which

is an irreducible 3-manifold V_i with one open ball removed. \hat{V}_* has one boundary sphere S_i for each V_i , $i = 1, \dots, k$, and in addition it has 2ℓ more spheres, two for each non-separating S_i of \mathcal{S} . If W has ℓ $S^2 \times S^1$ prime summands, and k irreducible summands, then there are ℓ non-separating S_i 's in \mathcal{S} and k separating spheres S_i in \mathcal{S} . The symbol \mathcal{S} denotes the set or the union of the spheres of \mathcal{S} .

We denote by \mathcal{S}' the set of duplicates of spheres of \mathcal{S} appearing on $\partial\hat{V}_*$. Choose an orientation for \hat{V}_* . Each sphere of \mathcal{S}' corresponds either to a separating sphere S , which obtains an orientation from \hat{V}_* , or it corresponds to a non-separating sphere of \mathcal{S} with an orientation.

We construct a *canonical compression body* associated to the 3-manifold W together with a symmetric system \mathcal{S} by thickening to replace W by $W \times I$, then attaching a 3-handle along $S_i \times 0$ for each $S_i \in \mathcal{S}$ to obtain a 4-manifold with boundary. Thus for each 3-handle attached to $S_i \times 0$, we have a 3-ball E_i in the 4-manifold with $\partial E_i = S_i$. The boundary of the resulting 4-manifold consists of W , one 3-sphere V_* , and the disjoint union of the irreducible non-sphere V_i 's. We cap the 3-sphere V_* with a 4-ball to obtain the *canonical compression body* Q associated to W marked by \mathcal{S} . Note that the exterior boundary of Q is W and the interior boundary is $V = \sqcup V_i$.

For each non-separating sphere S in \mathcal{S} we have an *associated separating sphere* $C(S)$, where $C(S)$ lies in \hat{V}_* and separates a 3-holed ball containing the duplicates of S coming from cutting on S . Also, we define $C(E)$ to be the obvious 3-ball in Q bounded by $C(S)$.

We have defined the canonical compression body as an object associated to the 3-manifold W *together with* the system \mathcal{S} . This will be our working definition. Later, we prove the uniqueness result of Proposition 1.6, which allows us to view the canonical compression body as being associated to W .

In some ways, the dual construction of the canonical compression body may be easier to understand: Let V_i , $i = 1 \dots k$ be the irreducible summands of W . Begin with the disjoint union of the $V_i \times I$ together with a 4-ball Z . For each i , identify a 3-ball E'_i in $V_i \times 1$ with a 3-ball E''_i in ∂Z to obtain a ball E_i in the quotient space. Then, for each $S^2 \times S^1$ summand of W identify a 3-ball E'_j in ∂Z with another 3-ball E''_j in ∂Z to obtain a disc E_j in the quotient. (Of course new E'_j 's and E''_j 's are chosen to be disjoint in ∂Z from previous ones.) The result of all the identifications is the canonical compression body Q for W with balls E_i properly embedded. We denote by \mathcal{E} the union of E_i 's. The system of spheres $\mathcal{S} = \partial\mathcal{E}$ is a symmetric system.

We now describe uniqueness for compression bodies associated to a 3-manifold W with a symmetric system.

Proposition 4.2. *If the canonical compression bodies (Q_1, V_1) and (Q_2, V_2) are associated to a compact 3-manifold W with symmetric systems \mathcal{S}_1 and \mathcal{S}_2 respectively, and $f : W \rightarrow W$ is any automorphism with $f(\mathcal{S}_1) = \mathcal{S}_2$, then there is a homeomorphism $g : (Q_1, V_1) \rightarrow (Q_2, V_2)$ with $g|_W = f$. Further, $g|_{V_1}$ is determined up isotopy.*

In particular, if (Q_1, V_1) and (Q_2, V_2) are canonical compression bodies associated to W with the system \mathcal{S} , then there is a homeomorphism $g : (Q_1, V_1) \rightarrow (Q_2, V_2)$ with $g|_W = \text{id}$. Further, $g|_{V_1}$ is determined up to isotopy.

Proof. Starting with $W \times I$, if we attach 3-handles along the spheres of $\mathcal{S}_1 \times 0$, then cap with a 4-ball we obtain Q_1 . Similarly we obtain Q_2 from $W \times I$ and $\mathcal{S}_2 \times 0$. Starting with $f \times \text{id} : W \times I \rightarrow W \times I$, we can then construct the homeomorphism g by mapping each 3-handle attached on a sphere of \mathcal{S}_1 to a 3-handle attached to the image sphere in \mathcal{S}_2 . Finally we map the 4-ball in Q_1 to the 4-ball in Q_2 . This yields g with $g|_W = f$. The uniqueness up to isotopy of $g|_{V_1}$ follows from Theorem 1.4.

Letting $\mathcal{S}_1 = \mathcal{S}_2$ and $f = \text{id}$, we get the second statement. \square

Lemma 4.3. *The canonical compression body (Q, V) associated to a compact 3-manifold W with symmetric system \mathcal{S} has the property that every 2-sphere $S \hookrightarrow W$ bounds a 3-ball properly embedded in Q .*

Proof. Make S transverse to \mathcal{S} and consider a curve of intersection innermost in S and bounding a disc D whose interior is disjoint from \mathcal{S} . If D lies in a holed irreducible \hat{V}_i , $i \geq 1$, then we can isotope to reduce the number of curves of intersection. If D lies in \hat{V}_* , then D union a disc D' in \mathcal{S} forms a sphere bounding a 3-ball in Q . (The 3-ball is embedded in the 4-ball Z .) We replace S by $S' = S \cup D' - D$, which can be pushed slightly to have fewer intersections with \mathcal{S} than S . Now if S' bounds a 3-ball, so does S , so by induction on the number of circles of intersection of S with \mathcal{S} , we have proved the lemma if we can start the induction by showing that a 2-sphere S whose intersection with \mathcal{S} is empty must bound a ball in Q . If such a 2-sphere S lies in \hat{V}_* , then it is obvious that it bounds a 3-ball in Z . If it lies in \hat{V}_i for some i , then it must be inessential or isotopic to S_i , hence it also bounds a ball (isotopic to E_i if S is isotopic to S_i). \square

Definition 4.4. Let W be a reducible 3-manifold. Let S be an essential sphere in W from a symmetric system \mathcal{S} . Or let S be a sphere associated to a non-separating sphere in \mathcal{S} . An \mathcal{S} -slide automorphism of W is an automorphism obtained by cutting on S , capping one of the resulting boundary spheres of $W|S$ with a ball B to obtain W' , isotoping the ball along a simple closed curve γ in W' returning the ball to itself, extending the isotopy to W' , removing the interior of the ball, and regluing the two boundary spheres of the manifold thus obtained from W' . We emphasize that when S separates a holed irreducible summand, either that irreducible summand can be replaced by a ball or the remainder of the manifold can be replaced by a ball. An \mathcal{S} -slide automorphism on a sphere S can be realized as the boundary of a \mathcal{E} -slide automorphism of $Q \text{ rel } V$: cut Q on the ball B bounded by S (B is either an E in \mathcal{E} or it is a ball $C(E)$ associated to a non-separating ball E in \mathcal{E}) to obtain a compression body Q' with two spots B' and B'' , duplicate copies of B . Slide the duplicate B' in $\partial_e Q'$ along a closed curve γ in $\partial_e Q'$, extending the isotopy to Q' . Then reglue B' to B'' to recover the compression body Q .

There is another kind of automorphism, called an \mathcal{S} -*interchanging slide* defined by cutting W on two separating spheres S_1 and S_2 , either both in \mathcal{S} cutting off holed irreducible summands \hat{V}_1 and \hat{V}_2 from the remainder \hat{W}_0 of W , or both spheres associated to different non-separating S in \mathcal{S} and cutting off holed $S^2 \times S^1$ summands \hat{V}_1 and \hat{V}_2 from W . Cap both boundary spheres in \hat{W}_0 and slide the capping balls along two paths from one ball to the other, interchanging them. Then reglue. The interchanging slide can also be realized as the boundary of an \mathcal{E} -*interchanging slide* of Q rel V . We will need to distinguish \mathcal{S} -*interchanging slides* of $S^2 \times S^1$ summands.

We define a further type of automorphism of W or Q . It is called an \mathcal{S} -*spin*. Let S be a non-separating sphere in \mathcal{S} . We do a half Dehn twist on the separating sphere $C(S)$ to map S to itself with the opposite orientation. Again, this spin can be realized as a half Dehn twist on the 3-ball $C(E)$ bounded by $C(S)$ in Q , which is an automorphism of Q rel V .

Finally, we consider Dehn twists on spheres of \mathcal{S} . These are well known to have order at most two in the mapping class group, and we call them \mathcal{S} -*Dehn twists*. These also are boundaries of \mathcal{E} -*Dehn twists*, automorphisms of Q rel V .

Remark 4.5. Slides, spins, Dehn twists on non-separating spheres, and interchanging slides of $S^2 \times S^1$ summands all have the property that they extend over Q as automorphisms rel V , so they represent elements of $\mathcal{H}_d(W)$.

We have defined \mathcal{S} -slides, and other automorphisms, in terms of the system \mathcal{S} in order to be able to describe the corresponding \mathcal{E} -slides in the canonical maximal compression body Q associated to W with the marking system \mathcal{S} . If we have not chosen a system \mathcal{S} with the associated canonical compression bodies, there are obvious definitions of *slides* (or interchanging slides, Dehn twists, or spins), without the requirement that the essential spheres involved be in \mathcal{S} or be associated to spheres in \mathcal{S} .

Definition 4.6. Let \mathcal{S}_1 and \mathcal{S}_2 be two symmetric systems for a 3-manifold W and let \mathcal{S}'_1 and \mathcal{S}'_2 denote the corresponding sets of duplicates appearing as boundary spheres of the holed spheres \hat{V}_{1*} and \hat{V}_{2*} , which are components of $W|\mathcal{S}_1$ and $W|\mathcal{S}_2$ respectively. Then an *allowable assignment* is a bijection $a : \mathcal{S}'_1 \rightarrow \mathcal{S}'_2$ such that a assigns to a duplicate of a non-separating sphere S , another duplicate of a non-separating sphere $a(S)$, and it assigns to the duplicate in \mathcal{S}'_1 of a separating sphere of \mathcal{S}_1 a duplicate in \mathcal{S}'_2 of a separating sphere of \mathcal{S}_2 cutting off a homeomorphic holed irreducible summand.

If an automorphism $f : W \rightarrow W$ satisfies $f(\mathcal{S}_1) = \mathcal{S}_2$, then f induces an allowable assignment $a : \mathcal{S}'_1 \rightarrow \mathcal{S}'_2$.

The following is closely related to results due to E. César de Sá [6], see also M. Scharlemann in Appendix A of [3], and D. McCullough in [12] (p. 69).

Lemma 4.7. *Suppose \mathcal{S} is a symmetric system for W and Q the corresponding compression body. Given any symmetric system $\mathcal{S}^\#$ for W , and an allowable assignment*

$a : \mathcal{S}' \rightarrow \mathcal{S}^\sharp$, there is a composition f of \mathcal{S} -slides, \mathcal{S} -slide interchanges, and \mathcal{S} -spins such that $f(\mathcal{S}) = \mathcal{S}^\sharp$, respecting orientation and inducing the map $a : \mathcal{S}' \rightarrow \mathcal{S}^\sharp$. The composition f extends over Q as a composition of \mathcal{E} -slides, \mathcal{E} -slide interchanges and \mathcal{E} -spins, all of which are automorphisms of $Q \text{ rel } V$.

Proof. W cut on the system \mathcal{S} is a manifold with components \hat{V}_i and \hat{V}_* ; W cut on the system \mathcal{S}^\sharp is a manifold with components \hat{V}_i^\sharp and \hat{V}_*^\sharp . Make \mathcal{S}^\sharp transverse to \mathcal{S} and consider a curve α of intersection innermost on \mathcal{S}^\sharp and bounding a disc D^\sharp in \mathcal{S}^\sharp . The curve α also bounds a disc D in \mathcal{S} ; in fact, there are two choices for D . If for either of these choices $D \cup D^\sharp$ bounds a ball, then we can isotope \mathcal{S}^\sharp to eliminate some curves of intersection. If D^\sharp is contained in a \hat{V}_i , $i \geq 1$, so that \hat{V}_i is an irreducible 3-manifold with one hole, then one of the choices of D must give $D \cup D^\sharp$ bounding a ball, so we may assume now that $D^\sharp \subset \hat{V}_*$, the holed 3-sphere. We can also assume that for both choices of D , $D \cup D^\sharp$ is a sphere which (when pushed away from $\partial \hat{V}_*$) cuts \hat{V}_* into two balls-with-holes, neither of which is a ball. Now we perform slides to move boundary spheres of \hat{V}_* contained in a chosen ball-with-holes bounded by $D \cup D^\sharp$ out of that ball-with-holes. That is, for each boundary sphere of \hat{V}_* in the ball-with-holes, perform a slide by cutting W on the sphere, capping to replace \hat{V}_i with a ball, sliding, and regluing. One must use a slide path disjoint from \mathcal{S}^\sharp , close to \mathcal{S}^\sharp , to return to a different component of $\hat{V}_* - \mathcal{S}^\sharp$, but this of course does not yield a closed slide path. To form a closed slide path, it is necessary to isotope $\mathcal{S}^\sharp \cap V_*$ rel ∂V_* . Seeing the existence of such a slide path requires a little thought; the key is to follow \mathcal{S}^\sharp without intersecting it. After performing slides to remove all holes of \hat{V}_* in the holed ball in V_* bounded by $D^\sharp \cup D$, we can finally isotope D^\sharp to D , and beyond, to eliminate curves of intersection in $\mathcal{S} \cap \mathcal{S}^\sharp$. Repeating, we can apply slide automorphisms and isotopies of \mathcal{S}^\sharp to eliminate all curves of intersection.

We know that \mathcal{S}^\sharp also cuts W into manifolds \hat{V}_i^\sharp homeomorphic to \hat{V}_i , and a holed sphere \hat{V}_*^\sharp and so we conclude that each sphere S of \mathcal{S} bounding \hat{V}_i , which is an irreducible manifold with an open ball removed, is isotopic to a sphere S^\sharp in \mathcal{S}^\sharp bounding \hat{V}_i^\sharp which is also an irreducible manifold with an open ball removed. Clearly also the \hat{V}_i bounded by S is homeomorphic to the \hat{V}_i^\sharp . Further isotopy therefore makes it possible to assume $f(\mathcal{S}) = \mathcal{S}^\sharp$. At this point, however, f does not necessarily assign each $S' \in \mathcal{S}'$ to the desired $a(S')$. This can be achieved using interchanging slides and/or spins. The interchanging slides are used to permute spheres of \mathcal{S}^\sharp ; the spins are used to reverse the orientation of a non-separating $S^\sharp \in \mathcal{S}^\sharp$, which switches two spheres of \mathcal{S}^\sharp .

Clearly we have constructed f as a composition of automorphisms which extend over Q as \mathcal{E} -slides, \mathcal{E} -slide interchanges and \mathcal{E} -spins, all of which are automorphisms of $Q \text{ rel } V$, hence f also extends as an automorphism of $Q \text{ rel } V$. \square

Before proving Proposition 1.7, we mention some known results we will need. The first result says that an automorphism of a 3-sphere with r holes which maps each boundary sphere to itself is isotopic to the identity. The second result concerns an

automorphism of an arbitrary 3-manifold: If it is isotopic to the identity and the isotopy moves the point along a null-homotopic closed path in the 3-manifold, then the isotopy can be replaced by an isotopy fixing the point at all times. These facts can be proved by applying a result of R. Palais, [14],[11].

Proof of Proposition 1.7. Let $g : W \rightarrow W$ be an automorphism. Then by Lemma 4.7, we conclude that there is an discrepant automorphism $f : (Q, V) \rightarrow (Q, V)$ mapping \mathcal{S} to $g(\mathcal{S})$ according to the assignment $\mathcal{S}' \rightarrow g(\mathcal{S})'$ induced by g . Thus $f^{-1} \circ g$ is the identity on \mathcal{S} , preserving orientation. Now we construct Q by attaching 3-handles to $W \times 0 \subset W \times I$, a handle attached to each sphere of \mathcal{S} . The automorphisms $f^{-1} \circ g$ and f clearly both extend over Q , hence so does g . This shows $\mathcal{H}(W) = \mathcal{H}_x(W)$.

Showing that $\mathcal{H}_d(W)$ has the indicated generators is more subtle. Suppose we are given $g : W \rightarrow W$ with the property that g extends to Q as an automorphism rel V . We also use g to denote the extended automorphism. We apply the method of proof in Lemma 4.7 to find an discrepant automorphism f , a composition of slides of holed irreducible summands \hat{V}_i , interchanging slides of $S^2 \times S^1$ summands and spins of $S^2 \times S^1$ to make $f \circ g(\mathcal{S})$ coincide with \mathcal{S} , preserving orientation. Note that f extends over Q , and that will be the case below as well, as we modify f . Thus we can regard f as an automorphism of Q . Notice also that in the above we do not need interchanging slides of irreducible summands, since g is the identity on V , hence maps each V_i to itself.

At this point $f \circ g$ maps each \hat{V}_i to itself, and of course it maps \hat{V}_* to itself, also mapping each boundary sphere of \hat{V}_* to itself. An automorphism of a holed sphere \hat{V}_* which maps each boundary sphere to itself is isotopic to the identity, as we mentioned above. However, we must consider this as an automorphism rel the boundary spheres, and so it may be a composition of twists on boundary spheres. On $W - \cup_i \text{int}(\hat{V}_i)$ (a holed connected sum of $S^2 \times S^1$'s) this yields some twists on non-separating spheres of \mathcal{S} corresponding to $S^2 \times S^1$ summands. Precomposing these twists with f , we can now assume that $f \circ g$ is the identity on $W - \cup_i \text{int}(\hat{V}_i)$.

It remains to analyze $f \circ g$ restricted to \hat{V}_i . We do this by considering the restriction of $f \circ g$ to $V_i \times I$, noting that we should regard this automorphism as an automorphism of the triple $(V_i \times I, E'_i, V_i \times 0)$, where E'_i is a duplicate of the disc E_i cutting $V_i \times I$ from Q . The product gives a homotopy $h_t = f \circ g|_{V_i \times t}$ from $h_0 = f \circ g|_{V_i \times 0} = \text{id}$ to $h_1 = f \circ g|_{V_i \times 1}$, where we regard these maps as being defined on V_i . Since V_i is an irreducible 3-manifold, we conclude as in the proof of Theorem 1.4, that h_0 is isotopic to h_1 via an isotopy h_t . From this isotopy, we obtain a path $\gamma(t) = h_t(p)$ where p is a point in E'_i when $V_i \times 1$ is identified with V_i . Consider a slide s along $\bar{\gamma}$: Cut Q on E_i , then slide the duplicate E'_i along the path $\bar{\gamma}$, extending the isotopy as usual. By construction, s is isotopic to the identity via s_t , $s_0 = \text{id}$, $s_1 = s$, and $s_t(p)$ is the path $\bar{\gamma}$. Combining the isotopies h_t and s_t , we see that $(s \circ f \circ g)|_{V_i \times 1}$ is isotopic to the identity via an isotopy r_t with $r_t(p)$ tracing the path $\gamma\bar{\gamma}$. This isotopy can be replaced by an isotopy fixing p or E'_i . Thus after replacing f by $s \circ f$, we have $f \circ g$ isotopic to the identity on \hat{V}_i . Doing the same thing for every i , we finally obtain f

so that $f \circ g$ is isotopic to the identity on W . This shows that g is a composition of the inverses of the generators used to construct f . \square

Now we can give a proof that the canonical compression body is unique in a stronger sense.

Proof of Proposition 1.6. We suppose (Q_1, V_1) is the canonical compression body associated to W with symmetric system \mathcal{S}_1 and (Q_2, V_2) is the canonical compression body associated to W with symmetric system \mathcal{S}_2 . From Proposition 4.2 we obtain a homeomorphism $h : (Q_1, V_1) \rightarrow (Q_2, V_2)$ such that $h(\mathcal{S}_1) = \mathcal{S}_2$. We are given a homeomorphism $v : V_1 \rightarrow V_2$. From the exact sequence of Theorem 1.5, we know that there is an automorphism $k : (Q_2, V_2) \rightarrow (Q_2, V_2)$ which extends the automorphism $v \circ (h^{-1}|_{V_2}) : V_2 \rightarrow V_2$. Then $g = k \circ h$ is the required homeomorphism $g : (Q_1, V_1) \rightarrow (Q_2, V_2)$ with the property that $g|_{V_1} = v$. \square

5 Spotted Manifolds.

In this section we present a quick “compression body point of view” on automorphisms of manifolds with sphere boundary components, or “spotted manifolds.”

Suppose W is an m -manifold, $m = 2$ or $m = 3$, with some sphere boundary components. For such a manifold, there is a canonical collection \mathcal{S} of essential spheres, namely a collection of spheres isotopic to the boundary spheres. We can construct the compression body associated to this collection of spheres, and we obtain an $n = m + 1$ dimensional compression body Q whose interior boundary consists of m -balls, one for each sphere of \mathcal{S} , and a closed manifold V_0 , which is the manifold obtained by capping the sphere boundary components. We denote the union of the m -balls by $P = \cup P_i$ where each P_i is a ball. Thus the interior boundary is $V = V_0 \cup P$. See Figure 2.

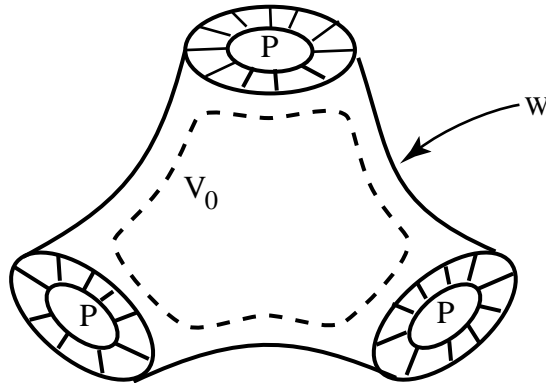


Figure 2: Spotted product.

Suppose V_0 has universal cover which is either S^m or is contractible. Applying Theorem 1.5, we obtain the exact sequence

$$1 \rightarrow \mathcal{H}_d(W) \rightarrow \mathcal{H}_x(W) \rightarrow \mathcal{H}(V) \rightarrow 1.$$

It remains to interpret the sequence. Notice that we can view Q as a *spotted product* $(V_0 \times I, P)$, where P is a collection of balls in $V_0 \times 1$. However, in our context we have a buffer zone of the form $S^{m-1} \times I \subset A$ separating each P_i from W . We refer to W as a holed manifold, since it is obtained from the manifold V_0 by removing open balls. $\mathcal{H}_d(W)$ is the mapping class group of automorphisms which extend to Q as automorphisms rel $V = V_0 \cup P$. Either directly, or using Theorem 1.4, we see that the elements of $\mathcal{H}_d(W)$ are automorphisms $f : W \rightarrow W$ such that the capped automorphism f_c is homotopic, hence isotopic to the identity.

The mapping class group $\mathcal{H}_x(W)$ can be identified with $\mathcal{H}(W)$. This is because any sphere of \mathbb{S} is mapped up to isotopy by any element $f \in \mathcal{H}(W)$ to another sphere of \mathbb{S} , hence f extends to Q . Finally $\mathcal{H}(V)$ can be identified with $\mathcal{H}(V_0) \times \mathcal{H}(P)$, where $\mathcal{H}(P)$ is clearly just a permutation group. We obtain the exact sequence

$$1 \rightarrow \mathcal{H}_d(W) \rightarrow \mathcal{H}(W) \rightarrow \mathcal{H}(V_0) \times \mathcal{H}(P) \rightarrow 1,$$

and we have proved Theorem 1.9.

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